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A FIXED UPPER BOUND ON THE NUMBER OF LAYERS IN OPTIMAL COMPOSITE SHEETS

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Abstract—This paper establishes a fixed upper limit of five on the number of layers required in an optimal plane-stress linear composite structure. The limit is independent of the complexity of the finite elements in terms of which the structure is idealised, the number of alternative loading conditions or the number and form of the constraints on deflection and stress, except that the latter must be expressible as functions of the deflections and fibre-angles only.

The limit arises as a consequences of a necessary condition satisfied by each individual finite element within an optimal design. This condition is shown to be that the thicknesses of layers in the element must be such as to solve an associated Linear Programming problem with, at most, five independent equality constraints. The limit follows from the fact that the form of the variation of layer stiffness with fibre angle depends on five basic functions.

NOTATION

The following is the main notation used; additional symbols will be defined as they are introduced.

- t_j^i the thickness of the jth layer in the ith element
- θ_j^i the fibre angle for the jth layer in the ith element
- $\mathbf{t}, \boldsymbol{\theta}$ the arrays of the above sets of variables
- L_i the number of layers in the ith finite element
- A_i the in-plane area, weighted if necessary to reflect cost rather than volume, of the ith finite element
- F^j a general real function of the deflections
- S^1 a stress constraint which is a general real function of deflections and fibre angles only
- ∂^k the N_d-vector of nodal deflections under the kth alternative load vector
- $\mathbf{P}^k$ load vector k, of dimension N_d
- N_f the number of deflection constraints for each alternative load case
- N_p the number of alternative load cases
- N_s the number of stress constraints for each alternative load case
- N_e the number of finite elements
- N_d the number of degrees of freedom of the structure
- H^i the allowable set of fibre angles in the ith finite element. Usually this denotes the range $(0, \pi)$ radians.

INTRODUCTION

The optimal design of structures composed of composite materials such as fibre-reinforced plastics presents a considerable challenge to numerical optimisers. Such structures involve several additional degrees of difficulty as compared with the relatively well-explored field of isotropic structures. One approach which has been suggested by the author involves formulating the problem in such a way that the variables whose optimal values are directly sought are not the design variables (thicknesses, fibre angles, layer numbers) but the deflections at the nodes from which these quantities can then be inferred [1, 2].

This approach, devised mainly as a computational tool, has provided some surprisingly simple and general insights into the necessary conditions for optimality of structures, extending beyond the field of composites [3]. In this paper these techniques will be used to establish a fixed, and quite low, upper bound on the number of layers necessary in an optimal multilaminar composite. The limit will be shown to result from the nature of multilaminar sheets composed of layers of orthogonal materials, in particular from the form of the stiffness coefficients as functions of fibre angle.

In the first section the optimisation problem will be defined, followed in Section 2 by a description of the significant properties of the material. Section 3 states and proves the upper bound, and is followed in Section 4 by a brief discussion of some of the implications.

1. THE PROBLEM

We are concerned with the characteristics of minimum volume structures whose elements are under states of plane stress. These elements are composed of multiple layers of a linearly

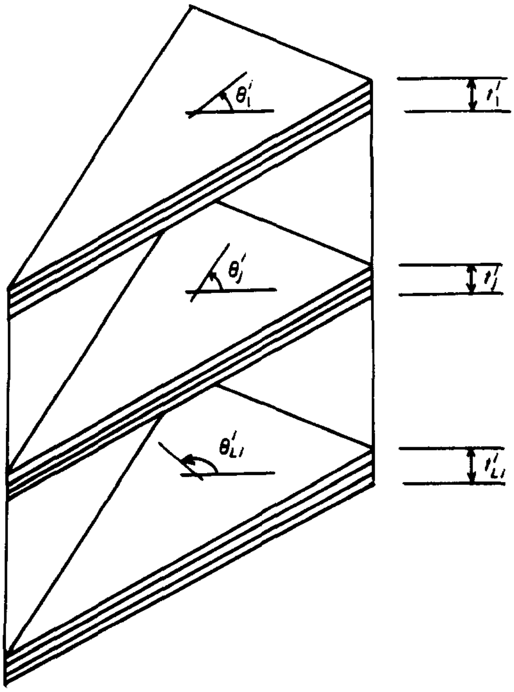

Fig. 1.

elastic orthotropic material; at any point on the structure each layer is distinguished from the others at that point by having a different angle (relative to some datum) for its axis of maximum strength and stiffness (fibre axis). The optimum structure is therefore that which has the minimum value of some linear function of the thicknesses subject to constraints on stresses and deflections.

It will be assumed that the structure has been divided into a number, N_e, of two-dimensional finite elements connected at nodes; the geometry of such elements and the form of the strain variation prescribed within them are arbitrary. The optimal structure, then, is that which has optimal values for the number of layers in each element, their fibre angles and their thicknesses. Figure 1 shows these quantities for a triangular element. The optimisation problem is as follows:

$$\min_{\mathbf{t},\boldsymbol{\theta},\mathbf{L}} W \equiv \sum_{i=1}^{N_e} A_i \sum_{j=1}^{L_i} t_j^i \qquad 1(\text{i})$$

$$\text{s.t.} \quad F^j(\boldsymbol{\delta}^k) \leq 0 \qquad j = 1,2\ldots N_f \qquad 1(\text{ii})$$

$$S^l(\boldsymbol{\delta}^k, \boldsymbol{\theta}) \leq 0 \qquad K = 1,2\ldots N_p \qquad 1(\text{iii})$$

$$K(\mathbf{t},\boldsymbol{\theta},\mathbf{L})\boldsymbol{\delta}^k = \mathbf{P}^k \qquad l = 1,2\ldots N_s \qquad 1(\text{iv})$$

$$t_j^i \geq 0 \qquad 1(\text{v})$$

$$\theta_j^i \epsilon H^i \qquad i = 1,2\ldots N_e \qquad 1(\text{vi})$$

$$L_i \text{ integer} \qquad j = 1,2\ldots L_i \ . \qquad 1(\text{vii})$$

2. THE MATERIAL

Tsai and Pagano ([4], quoted by Hadcock in [5]) give the following relationship between in-plane stress and strain for fibre-reinforced composites:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & Q_{16}\\ Q_{21} & Q_{22} & Q_{26}\\ Q_{61} & Q_{62} & Q_{66}\end{bmatrix}\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_{xy}\end{bmatrix} \tag{2}$$

(where the notation of [5] has been adhered to for the Q subscripts.)

The Q_{ij} are the following functions of fibre angle:

$$\begin{aligned}
Q_{11} &= 3U_1 + U_2 + U_3\mathrm{Cos}\,2\theta + U_4\mathrm{Cos}\,4\theta\\
Q_{22} &= 3U_1 + U_2 - U_3\mathrm{Cos}\,2\theta + U_4\mathrm{Cos}\,4\theta\\
Q_{21} &= Q_{12} = U_1 - U_2 - U_4\mathrm{Cos}\,4\theta\\
Q_{66} &= U_1 + U_2 - U_4\mathrm{Cos}\,4\theta\\
Q_{61} &= Q_{16} = \tfrac{1}{2}U_3\mathrm{Sin}\,2\theta + U_4\mathrm{Sin}\,4\theta\\
Q_{62} &= Q_{26} = \tfrac{1}{2}U_3\,\mathrm{Sin}\,2\theta - U_4\,\mathrm{Sin}\,4\theta\\
U_1 &\equiv (E_{11} + E_{22} + \nu_{21}E_{11} + \nu_{12}E_{22})/8\psi\\
U_2 &\equiv (\psi G_{12} - \tfrac{1}{2}(\nu_{21}E_{11} + \nu_{12}E_{22}))/2\psi\\
U_3 &\equiv (E_{11} - E_{22})/2\psi\\
U_4 &\equiv (E_{11} + E_{22} - (\nu_{21}E_{11} + \nu_{12}E_{22}) - 4\psi G_{12})/8\psi.
\end{aligned}$$

E_{11} and E_{22} are respectively the longitudinal and transverse moduli of elasticity of the layer. G_{12} is the in-plane shear modulus; v_{12} is the ratio of transverse-to-longitudinal strain under longitudinal stress; $\nu_{12}E_{22} = \nu_{21}E_{12}$; $\psi = 1 - \nu_{12}\nu_{21}$. Figure 2 shows the conventions used for these quantities.

It is clear that relationship (2) can be rewritten in the following form:

$$\boldsymbol{\sigma} = (\mathbf{q}_0 + \mathbf{q}_1\cos 4\theta + \mathbf{q}_2\sin 4\theta + \mathbf{q}_3\cos 2\theta + \mathbf{q}_4\sin 2\theta)\boldsymbol{\epsilon} \tag{3}$$

$$q_0 \equiv \begin{bmatrix}3U_1 + U_2 & U_1 - U_2 & 0\\ U_1 - U_2 & 3U_1 + U_2 & 0\\ 0 & 0 & U_1 + U_2\end{bmatrix}$$

$$q_1 \equiv \begin{bmatrix}U_4 & -U_4 & 0\\ -U_4 & U_4 & 0\\ 0 & 0 & -U_4\end{bmatrix}$$

$$q_2 \equiv \begin{bmatrix}0 & 0 & U_4\\ 0 & 0 & -U_4\\ U_4 & -U_4 & 0\end{bmatrix}$$

$$q_3 \equiv \begin{bmatrix}U_3 & 0 & 0\\ 0 & -U_3 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$q_4 \equiv \begin{bmatrix}0 & 0 & 1/2\,U_3\\ 0 & 0 & 1/2\,U_3\\ \frac{1}{2}U_3 & \frac{1}{2}U_3 & 0\end{bmatrix}.$$

Since the stiffness matrix of a layer is linear in the stiffness coefficients of the material, eqn 3

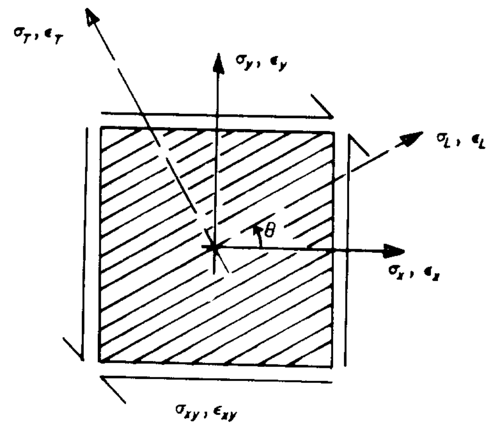

Fig. 2.

implies that such a stiffness matrix for a layer of thickness t^i and fibre angle θ^i in the i'th finite element will be of the general form:

$$\mathbf{k}^i(\theta^i) = t^i(\mathbf{k}_0{}^i + \mathbf{k}_1{}^i \cos 4\theta^i + \ldots \mathbf{k}_4{}^i \sin 2\theta). \tag{4}$$

This form is of course independent of the actual finite elements used in the idealisation. It is a property of any structural element consisting of a single layer of orthotropic material, and upon it rests the main result to be established in the next section.

3. AN UPPER BOUND THEOREM

Let D^* be an optimum design in the sense of Problem (1) and let $\boldsymbol{\delta}^{j*}$ be the deflections of this design under the load vector $\mathbf{P}_j$. Two lemmas will be established.

Lemma 1: Let a realisable perturbation be one which maintains non-negativity of t_j^i. Any realisable perturbation to the design D^* which leaves unaltered all deflections under each of the alternative load sets, and which does not change the fibre angles in any layer, is a feasible solution to Problem 1.

Proof: Since D^* is optimal it is also feasible. Any perturbation to it which leaves deflections unaltered clearly leaves it feasible with respect to constraints 1(ii) which are functions of these deflections alone. If, in addition, no new angles are introduced then the new design also remains feasible with respect to 1(iii), the stress constraints.

Lemma 2: Let $\mathbf{t}^{i*}$ be the vector of layer thicknesses, L_i^* in number, for the i'th finite element; $\boldsymbol{\theta}^{i*}$ is the corresponding vector of fibre angles. A necessary condition for D^* to be optimal is that, for the i'th element, the vector $\mathbf{t}^{i*}$ is the optimal solution to the following problem:

$$\min_{\mathbf{t}^i} W(\mathbf{t}^i) \equiv \sum_{j=1}^{L_i^*} A_i t_j^i \qquad 5(\text{i})$$

$$\text{s.t.} \quad \mathbf{K}^i(\mathbf{t}^i, \boldsymbol{\theta}^{i*})\boldsymbol{\delta}^{k*} = \mathbf{P}^{k,i} \quad k = 1,2 \ldots N_p \qquad 5(\text{ii})$$

$$\mathbf{t}^i \geqslant 0 \qquad 5(\text{iii})$$

where $\mathbf{K}^i$ is the total stiffness matrix of the element and $\mathbf{P}^{k,i}$ is the vector of nodal loads in the element under the k'th overall load vector $\mathbf{P}^k$.

Proof: Any thickness vector $\mathbf{t}^i$ which satisfies the equality constraints 5(ii) keeps both the deflection and nodal loads constant, and therefore allows the element to be treated in isolation from the rest of the structure. By Lemma 1 such a design is also feasible so long as all its elements are nonnegative. Let t^{i**} be the design which minimises 5(i) subject to 5(ii) and 5(iii), and assume $W^i(\mathbf{t}^{i**}) < W^i(\mathbf{t}^{i*})$. Since the rest of the structure is unaffected by substituting t^{i**} for $\mathbf{t}^{i*}$ in D^*, it follows that a design would have been found which was feasible but of lower function value than D^*. Therefore, if D^* is already optimal then $W^i(\mathbf{t}^{i**}) = W^i(\mathbf{t}^{i*})$ and, since W^i is linear in $\mathbf{t}^i$, $\mathbf{t}^{i**} = \mathbf{t}^{i*}$. This proves the lemma.

These lemmas provide the main results needed to prove the following theorem.

Theorem: Let N_d^i be the number of independent deformation modes of the i'th finite element in a structure which is optimal in the sense of Problem 1. Then the number of layers in that element will never exceed either five or the number $N_p(N_d^i - (N_p - 1)/2)$, whichever is the lower.

Proof: Consider in more detail the optimisation Problem 5 of Lemma 2. For simplicity in the first instance consider only one load case, $N_p = 1$. The stiffness matrix for the element is the sum of those of the individual layers, i.e.

$$\mathbf{k}^i = \sum_{j=1}^{L_i^*} \mathbf{k}^i(\theta_j^i)$$

and, using (4) to replace $\mathbf{k}^i(\theta_j^i)$:

$$\mathbf{k}^i = \sum_{j=1}^{L_i^*} t_j^i(\mathbf{k}_0{}^i + \mathbf{k}_1{}^i \cos 4\theta_j^i + \mathbf{k}_2{}^i \sin \theta_j^i$$
$$+ \mathbf{k}_3{}^i \cos 2\theta_j^i + \mathbf{k}_4{}^i \sin 2\theta_j^i)$$

Hence the equality constraints can be written (since $\boldsymbol{\delta}^{1*}$ is fixed) as:

$$\sum_{j=1}^{L_i^*} t_j^i \mathbf{p}_j^i = \mathbf{P}^{1,i}$$

Where $\mathbf{p}_j^i$ is the vector of nodal loads exerted by unit thickness of the i, j'th layer when subjected to a deflection $\boldsymbol{\delta}^{1*}$ and is given by:

$$\mathbf{p}_j^i \equiv \mathbf{k}_0{}^i\boldsymbol{\delta}^{1*} + (\mathbf{k}_1{}^i\boldsymbol{\delta}^{1*}) \cos 4\theta_j^i + \ldots (\mathbf{k}_4{}^i\boldsymbol{\delta}^{1*}) \sin 2\theta_j^i$$

Introduce the following notation:

$$\mathbf{B}^i \equiv [\mathbf{k}_0{}^i\boldsymbol{\delta}^{1*}, \mathbf{k}_1{}^i\boldsymbol{\delta}^{1*}, \ldots \mathbf{k}_4{}^i\boldsymbol{\delta}^{1*}]$$

and $\boldsymbol{\phi}_j^i$ is the 5-vector:

$$[1, \cos 4\theta_j^i, \sin 4\theta_j^i, \cos 2\theta_j^i, \sin 2\theta_j^i]'$$

so that:

$$\mathbf{p}_j^i = \mathbf{B}^i\boldsymbol{\phi}_j^i.$$

The complete equality constraint set can therefore be written:

$$B^i\Phi^i t^i = P^{1,i} \tag{6}$$

where $\boldsymbol{\Phi}^i$ is the $5 \times L_i^*$ matrix whose columns are the ϕ_j^i vectors. Thus, Problem 5 is a linear programming one in the t_j^i subject to the N_d equality constraints (6). Of course, not all these constraints are independent in general. Let $\mathbf{B}^{i\prime}$ and $\mathbf{P}^{1,i\prime}$ define the reduced set obtained by eliminating all equations corresponding to linearly dependent rows of B^i:

$$\mathbf{B}^{i\prime}\boldsymbol{\Phi}^i\mathbf{t}^i = \mathbf{P}^{1,i\prime} \quad . \tag{7}$$

The rank of these remaining equations is equal to the rank of $\mathbf{\Phi}^i$ or $\mathbf{B}^{i\prime}$, whichever is the less (e.g. Theorem 33,[6]). Now $B^{i\prime}$ has five columns, and $\mathbf{\Phi}^i$ five rows; the maximum rank of either is thus five. However, the rank of $B^{i\prime}$ cannot exceed N_d^i since that is the rank of $\mathbf{k}^i(\theta^i)$, so the number of independent equality constraints on the LP is min $(5, N_d^i)$.

Now consider the effect of increasing the number of alternative load cases. For each additional load we add the set of equations for the equilibrium of the i'th element:

$$\mathbf{K}^i\boldsymbol{\delta}^{k*} = \mathbf{P}^{k,i}$$

However, by the principle of virtual work each deflection and nodal load set must satisfy:

$$\mathbf{P}^{r,i\prime}\boldsymbol{\delta}^{s*} = \boldsymbol{\delta}^{r*\prime}\mathbf{P}^{s,i}$$

Hence, N_p load cases imply at most $N_p(N_d^i - (N_p - 1)/2)$ independent rows of $A^{i\prime}$, and $\mathbf{\Phi}^i$ is of course unaltered. The maximum rank of the equations becomes:

$$R = \min\{5, N_pN_d^i - N_p(N_p - 1)/2\}.$$

Hence, by Lemma 2, the vector $\mathbf{t}^{i*}$ solves a linear programming problem with R equality constraints; by the fundamental theory of Linear Programming, therefore, not more than R values of t_j^i may be nonzero. Q.E.D.

There are some results which follow from this theorem, or can be derived using slight variations of the proof.

Let a 'balanced doublet' be defined as a double layer, one half-thickness of which has a fibre angle θ and the other an angle of $-\theta$ relative to the datum axis. Define an "orthogonal doublet" as a double layer with the angle in one half thickness at right angles to that in the remaining half thickness.

Now consider two restricted forms of problem (1) obtained by applying one of the following constraints (i) the structure shall be composed of balanced doublets only; (ii) the structure shall consist of orthogonal doublets only. In either of these cases, the following corollary applies:

Corollary 1: The maximum number of balanced or orthogonal doublets in any element is three or $N_p(N_d^i - (N_p - 1)/2$, whichever is smaller.

Proof: Consider eqn (4). For the doublets defined above, some terms vanish in this expression. For the balanced doublet, terms in odd functions vanish, while for the orthogonal doublet terms in functions whose arguments are $2\theta^i$ vanish. Hence, in both of these cases, the expansion has only three terms and the corollary follows by the same reasoning as the Theorem.

Consider the case where the minimum thicknesses imposed are, for some arbitrary fibre angles, greater than zero. The following corollary is true.

Corollary 2: If some thicknesses are required to be greater than some nonzero lower limit, then the theorem applies to this case with the term 'number of layers' replaced by 'number of layers whose thicknesses are strictly greater than their lower limit'.

Proof: The corollary follows immediately from the proof of the theorem by replacing $\mathbf{t}^i$ by $(t^i - \mathbf{t}_0^i)$, where $\mathbf{t}_o^i$ is the vector of lower thickness limits.

Another form of constraint which might be included in Problem 1 is that the total thickness in the element must be greater than some positive number. This would have the effect of ensuring that the optimal design did not have gaps where finite elements had been left empty. The bound in this case is given by the following corollary.

Corollary 3: If a positive lower limit is imposed on the total thickness in any element, then the absolute upper limit on the number of layers given by the theorem increases from five to six.

Proof: In Lemma 2, an additional constraint must be added, say

$$\sum_{j=1}^{L_i} t_j^i \geq t_0^i$$

Then, in the proof of the theorem, the linear program also has an additional constraint and the corollary follows.

In the same way, the upper limit on the number of doublets is also increased by one in this case. Corollary 3 will apply in most practical cases.

4. CONCLUSION

The theorem proved above is very general in the sense that it applies to a wide range of optimal composites. In particular, the upper limit of five layers is not affected by the following:

(i) The number of alternative load cases;

(ii) The number or form of the deflection constraints;

(iii) The number or form of the stress constraints so long as these can be expressed as functions of strain and fibre angles only;

(iv) The size or complexity of the finite elements; evidently the limit applies to the continuum since this could be viewed as a single element of infinite complexity.

It is equally important to stress the main assumption underlying the result, which is that the equilibrium equation 1(iv) must involve membrane loads and displacements only. This means that only in-plane loads may be applied, and that any layer must be considered as being split into two of equal thickness, these being symmetrically disposed about the middle surface of the sheet. Such a provision would normally be made in constructing such sheets.

Finally, although the terminology of fibre-reinforced composites has been used throughout this note, the results obtained will of course apply to all structures composed of layers of orthotropic material which satisfy eqn (3).

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